Magnetic field generation from self-consistent collective neutrino-plasma interactions

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A Lagrangian formalism for self-consistent collective neutrino-plasma interactions is presented in which each neutrino species is described as a classical ideal fluid. The neutrino-plasma fluid equations are derived from a covariant relativistic variational principle in which finite-temperature effects are retained. This formalism is then used to investigate the generation of magnetic fields and the production of magnetic helicity as a result of collective neutrino-plasma interactions.

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I. INTRODUCTION

Photons, neutrinos, and plasmas are ubiquitous in the Universe [1,2]. During the early Universe, it is expected that photons and neutrinos interacted quite strongly with hot primordial plasmas [3]. Although photons and neutrinos decoupled from plasmas relatively early after the big bang [1,2], there are still conditions today where neutrino-plasma interactions might be important. For example, during a supernova explosion [4–6], intense neutrino fluxes are generated as result of the gravitational collapse of the stellar core. It is generally believed that the outgoing neutrino flux needs to transfer energy and momentum to the surrounding plasma in order to produce the observed explosion.

The self-consistent collective interaction between photons and plasmas is traditionally treated classically (i.e., without quantum-mechanical effects), where plasma particles are treated within either a fluid or a kinetic picture, while photons are described in terms of an electromagnetic field. For a self-consistent treatment of collective electromagnetic-plasma interactions (see Ref. [7], for example), one considers both the influence of electromagnetic fields on plasma dynamics and the generation of electromagnetic fields by plasma currents. The interaction between photons and neutrinos, on the other hand, requires a full quantum-mechanical treatment and has been the subject of recent interest [8].

Neutrino-plasma interactions involve charged and neutral currents associated with the weak force [9,10] (through the exchange of $W^{\pm}$ and Z^0 bosons, respectively). The collective interactions studied here apply to the case of intense neutrino fluxes. Discrete (i.e., collisional) neutrino-plasma interactions, on the other hand, involve scattering of individual particles; such discrete neutrino-plasma particle effects will be omitted in the present work.

The purpose of the present work is to investigate the self-consistent collective interaction between neutrinos and plasmas in the presence of electromagnetic fields. The inclusion of electromagnetic effects is a departure from conventional hydrodynamic models used in investigating neutrino interactions with astrophysical plasmas [5]. Here, we investigate the collective processes

$$\mathrm{EM}\rightarrow\sigma\rightarrow\nu \tag{1.1}$$

and

$$\nu\rightarrow\sigma\rightarrow\mathrm{EM}. \tag{1.2}$$

In the first process, the neutrino (ν) dynamics is influenced by an electromagnetic field (EM) with a plasma (σ) background acting as an intermediary, even though neutrinos are chargeless particles. In the second process, electromagnetic fields are generated as a result of plasma currents produced by neutrino ponderomotive effects. The problem of magnetic field generation associated with self-consistent collective neutrino-plasma interactions is thus investigated here within the context of the process (1.2).

A. Notation

In the present paper, the Latin subscript s refers to different components of the neutrino-plasma fluid: the subscript $s=\nu$ refers to neutrinos while the subscript $s=\sigma$ refers to components of the plasma other than photons and neutrinos. To avoid confusion, we use the Greek letters $\alpha,\beta,\ldots$ for Lorentz indices rather than traditional $\mu,\nu,\ldots$; for example, the flux four-vector is $J^{\alpha}=Nu^{\alpha}$, with proper density N (Lorentz scalar) and normalized four-velocity $u^{\alpha}=(u^0,\mathbf{u})$. In certain cases, objects with Lorentz indices may not be covariant; for instance, the fluid velocity $v^{\alpha}=u^{\alpha}/u^0$ is not covariant and the number density in a given frame $n=Nu^0$ is not a Lorentz scalar. The symbols in boldface are three-vectors while those in sans serif are four-dimensional tensors (such as F for the electromagnetic field strength $F_{\alpha\beta}$). The dot $\cdot$ describes the contraction of a Lorentz index or an inner product of two three-vectors if in boldface. Here, we employ the metric $g_{\alpha\beta}=\mathrm{diag}(1,-1,-1,-1)$ and, hence, $a\cdot b\equiv a^0b^0-\mathbf{a}\cdot\mathbf{b}$.

B. Neutrino descriptions for collective neutrino-plasma interactions

To study collective neutrino-plasma interactions, neutrinos can be described in terms of Dirac spinor fields [9–13], Klein-Gordon scalar fields [14,15], classical nonrelavistic fluids [16], or relativistic quasiparticles [17,18]. In all these descriptions, the interaction between neutrinos (of type ν) and plasma particles (of species σ) is described in terms of an effective weak-interaction charge $G_{\sigma\nu}$. In general, $G_{\sigma\nu}$ has the following property [11]:

$$G_{\sigma\nu} = -G_{\bar{\sigma}\nu} = -G_{\sigma\bar{\nu}} = G_{\bar{\sigma}\bar{\nu}}, \tag{1.3}$$

where σ ($\bar{\sigma}$) denotes a matter (antimatter) species and ν ($\bar{\nu}$) denotes a neutrino (antineutrino) species. The effective charge $G_{\sigma\nu}$ depends on the Fermi weak-interaction constant G_F ($\approx 9\times10^{-38}$ eV cm^3), the Weinberg angle θ_W ($\sin^2\theta_W \approx 0.23$ [10]), and the species σ and ν. For example, for neutrinos interacting with unpolarized electrons (e), protons (p), and neutrons (n), one finds [11]

$$G_{\sigma\nu} = \sqrt{2} G_F \, [\delta_{\sigma e}\delta_{\nu\nu_e} + (I_\sigma - 2Q_\sigma \sin^2\theta_W)], \tag{1.4}$$

where I_σ is the weak isotopic spin for particle species σ ($I_e = I_n = -1/2$ and $I_p = 1/2$) and $Q_\sigma \equiv q_\sigma/e$ is the normalized electric charge. Here, the first term in Eq. (1.4) is due to charged weak currents (and thus applies only to electrons and electron-neutrinos), while the remaining terms are due to neutral weak currents (and thus apply to all species).

To assist us in investigating self-consistent collective neutrino-plasma interactions in the present work, all neutrino and particle species are treated as ideal classical fluids. For this purpose, we proceed with the classical fluid limit for plasma particles in the Dirac description expressed in terms of the correspondence

$$\bar{\psi}_\sigma(\hat{\gamma}^\alpha/c)\psi_\sigma \rightarrow J^\alpha_\sigma \equiv (n_\sigma, \boldsymbol{J}_\sigma), \tag{1.5}$$

where ψ_σ is the Dirac spinor field for particle species σ (with $\hat{\gamma}^\alpha$ denoting Dirac matrices) while n_σ and $\boldsymbol{J}_\sigma \equiv n_\sigma \mathbf{v}_\sigma/c$ are the particle density and (normalized) particle flux for each plasma-fluid species σ in the laboratory reference frame, respectively. In this limit, the propagation of a neutrino test particle of type ν in a background plasma is determined by the effective potential [19]

$$V_\nu(\mathbf{x},\mathbf{v},t) \equiv \sum_\sigma G_{\sigma\nu}\left(n_\sigma(\mathbf{x},t) - \boldsymbol{J}_\sigma(\mathbf{x},t)\cdot\frac{\mathbf{v}}{c}\right), \tag{1.6}$$

where $(\mathbf{x},\mathbf{v})$ denote the neutrino's position and velocity. We note that neutrino propagation in matter is a topic at the heart of the problem of neutrino oscillations in matter [20–22] and the solar neutrino problem [23]. Although the term $\boldsymbol{J}_\sigma\cdot\mathbf{v}/c$ is a relativistic correction to n_σ in Eq. (1.6), we keep it for the following reason. For a primordial plasma with a single family of particles ($s=\sigma$) and antiparticles ($s=\bar{\sigma}$), we find from Eq. (1.3)

$$\sum_{s=\sigma,\bar{\sigma}} G_{s\nu} n_s = 0,$$
$$\sum_{s=\sigma,\bar{\sigma}} G_{s\nu}\boldsymbol{J}_s = G_{\sigma\nu}(\boldsymbol{J}_\sigma - \boldsymbol{J}_{\bar{\sigma}}), \tag{1.7}$$

and thus the effective neutrino potential (1.6) becomes $V_\nu = G_{\sigma\nu}(\boldsymbol{J}_{\bar{\sigma}} - \boldsymbol{J}_\sigma)\cdot\mathbf{v}/c$, for each $(\sigma,\bar{\sigma})$ family. Hence, keeping this relativistic correction is necessary for the description of collective neutrino interactions with a primordial plasma [24]. The model presented here therefore retains all relativistic effects associated with the neutrino and plasma fluids.

For a self-consistent description of collective neutrino-plasma interactions in which neutrino ponderomotive effects on the background medium are included, we now use a similar classical-fluid correspondence for the neutrinos. The propagation of a plasma test particle of species σ (with electric charge q_σ) in a background medium composed of a neutrino fluid of type ν and an electromagnetic field is determined by the potential

$$V_\sigma(\mathbf{x},\mathbf{v},t) \equiv \left(q_\sigma\,\phi(\mathbf{x},t) + \sum_\nu G_{\sigma\nu}\, n_\nu(\mathbf{x},t)\right) - \left(q_\sigma \mathbf{A}(\mathbf{x},t) + \sum_\nu G_{\sigma\nu}\boldsymbol{J}_\nu(\mathbf{x},t)\right)\cdot\frac{\mathbf{v}}{c}, \tag{1.8}$$

where n_ν and $\boldsymbol{J}_\nu \equiv n_\nu \mathbf{v}_\nu/c$ are the neutrino density and (normalized) neutrino flux in the laboratory reference frame, respectively, ϕ and $\mathbf{A}$ are the electromagnetic potentials, and $(\mathbf{x},\mathbf{v})$ denote the plasma particle's position and velocity. It is interesting to note how the right side of Eq. (1.8) links the electrostatic scalar potential ϕ and the neutrino density n_ν, on the one hand, and the magnetic vector potential $\mathbf{A}$ and the neutrino flux vector $\boldsymbol{J}_\nu$, on the other hand. We will henceforth refer to the approximation whereby $\boldsymbol{J}_\sigma$ and $\boldsymbol{J}_\nu$ are omitted in Eqs. (1.6) and (1.8) as the *weak* electrostatic (or nonrelativistic) approximation.

Although we assume that each neutrino flavor has a finite mass, this assumption is not crucial to the development of our model; see Sec. II for a discussion of neutrino-fluid dynamics for arbitrary neutrino masses. Furthermore we shall ignore all quantum-mechanical effects, including effects due to strong magnetic fields [25] (i.e., we assume $B/B_{\rm QM} \equiv \hbar\Omega_e/m_e c^2 \ll 1$, where $\Omega_e \equiv eB/m_e c$ is the electron gyrofrequency and $B_{\rm QM} \sim 4\times10^{13}$ G). Hence, although magnetic fields appear explicitly in our model, they are not considered strong enough to modify the form of the interaction potentials (1.6) and (1.8).

C. Magnetic field generation due to neutrino-plasma interactions

An important application of the process (1.2) involves the prospect of generating magnetic fields in an unmagnetized plasma as a result of collective neutrino-plasma interactions. This application may be of importance in investigating magnetogenesis in the early universe (e.g., see Ref. [2]). A similar process of magnetic field generation occurs in laser-plasma interactions whereby an intense laser pulse propagating in a nonuniform plasma generates a quasistatic magnetic field. This process was first studied theoretically [26–28] and was recently confirmed experimentally [29].

The generation of magnetic fields by collective neutrino-plasma interactions was first contemplated in the nonrelativistic (weak electrostatic) limit by Shukla *et al.* [30,31]. The covariant (relativistic) Lagrangian approach introduced by Brizard and Wurtele [15], however, revealed the presence of additional ponderomotive terms missing from previous analysis [14,30,31]. These additional ponderomotive terms involve the time derivative of the neutrino flux $\partial_t\boldsymbol{J}_\nu$ and the curl of the neutrino flux $\nabla\times\boldsymbol{J}_\nu$ (henceforth referred to as the neutrino-flux vorticity), which are shown here to lead to significantly different predictions regarding neutrino-induced magnetic field generation. In fact, we show that magnetic field generation due to neutrino-plasma interactions is not possible without these terms.

D. Organization

The remainder of this paper is organized as follows. In Sec. II, the Lagrangian formalism for ideal fluids is introduced. In Sec. III, a variational principle for collective neutrino-plasma interactions in the presence of an electromagnetic field is presented. This Lagrangian formalism is fully relativistic and covariant and can thus be generalized to include general relativistic effects (e.g., see Refs. [32,33]). In Sec. IV, the nonlinear neutrino-plasma fluid equations and the Maxwell equations for the electromagnetic field are derived. Through the Noether method [34–36], an exact energy-momentum conservation law is also derived and the process of energy-momentum transfer from the neutrinos to the electromagnetic field and the plasma is discussed. In Sec. V, magnetic field generation, magnetic helicity production, and magnetic equilibrium involving neutrino-plasma interactions are investigated. Here, we find that neutrino-flux vorticity ($\boldsymbol{\nabla}\times \boldsymbol{J}_\nu$) plays a fundamental role in all three processes. We summarize our work in Sec. VI and discuss future work.

II. LAGRANGIAN DENSITY FOR A FREE IDEAL FLUID

The present section is dedicated to the derivation of a suitable Lagrangian density for a free ideal fluid from an existing single-particle Lagrangian for a free particle of arbitrary mass (including zero). The difficulty with dealing with the case of free neutrinos as particles is that their mass may be zero. Since the relativistic Lagrangian L for a free single particle of mass m is [37]

$$L=-\,mc^2\,\gamma^{-1}\equiv -\,mc\left(\frac{dx^\alpha}{dt}\,\frac{dx_\alpha}{dt}\right)^{1/2}, \tag{2.1}$$

it is not obvious how to handle the limiting case of zero mass. This difficulty is resolved in [38] as follows (see also Ref. [2]).

A. Single-particle Lagrangian

Consider the primitive Lagrangian

$$L_p=\mathbf{p}\cdot\dot{\mathbf{x}}-E\,\dot{t}\equiv -\,p_\alpha c\,v^\alpha \tag{2.2}$$

for a particle of arbitrary rest mass m (including zero), where (x,p) are coordinates in the eight-dimensional phase space in which the particle moves and $\dot{x}^\alpha=(c,\mathbf{v})\equiv cv^\alpha$. Although the particle's space-time location $x^\alpha=(ct,\mathbf{x})$ is arbitrary, its four-momentum $p^\alpha=(E/c,\mathbf{p})$ is not, since the particle's physical motion is constrained to occur on the mass shell

$$p_\alpha p^\alpha=m^2c^2. \tag{2.3}$$

Here, $u^\alpha\equiv\gamma v^\alpha$ is the normalized four-velocity and $\gamma=(1+|\mathbf{u}|^2)^{1/2}$ is the relativistic factor.

Since the mass constraint (2.3) cannot be derived from the primitive Lagrangian (2.2), we explicitly introduce it by means of a Lagrange multiplier:

$$L_p\equiv -\,p_\alpha c\,v^\alpha-\frac{1}{2\lambda}(\,m^2c^4-p_\alpha p^\alpha\,c^2\,), \tag{2.4}$$

where λ^{-1} is the Lagrangian multiplier and the factor 1/2 is added for convenience. Since the Lagrangian (2.4) is independent of $\dot{p}_\alpha$, the Euler-Lagrange equation for p_α yields

$$\frac{\partial L_p}{\partial p_\alpha}=-\,cv^\alpha+\frac{p^\alpha c^2}{\lambda}\equiv 0, \tag{2.5}$$

from which we obtain

$$p^\alpha=\lambda\,v^\alpha/c. \tag{2.6}$$

Using the mass constraint (2.3) and the identity $v\cdot v\equiv\gamma^{-2}$, the relation (2.6) yields

$$\lambda=\gamma mc^2, \tag{2.7}$$

i.e., λ is the energy of a single particle of mass m.

If we now substitute Eq. (2.6) into the primitive Lagrangian (2.4) (i.e., by constraining the physical motion to take place on the mass shell), we find the physical Lagrangian

$$L(v;\lambda)\equiv L_p(x;p=\lambda v/c;\lambda)=-\frac{m^2c^4}{2\lambda}-\frac{\lambda}{2\gamma^2}. \tag{2.8}$$

This Lagrangian now depends only on v^α and λ (for a free particle, there is no space-time dependence in the Lagrangian). The Euler-Lagrange equation for λ now yields

$$\frac{\partial L}{\partial\lambda}=\frac{1}{2}\left(\frac{m^2c^4}{\lambda^2}-\frac{1}{\gamma^2}\right)\equiv 0, \tag{2.9}$$

which gives Eq. (2.7). Substituting Eq. (2.7) into Eq. (2.8) yields the standard Lagrangian (2.1).

For a massless particle, on the other hand, the condition (2.9) yields

$$\gamma^{-2}=v_\alpha v^\alpha\equiv 0, \tag{2.10}$$

which states that massless particles travel at the speed of light. Here, λ is still the massless particle's energy since Eq. (2.6) gives $p^0\equiv\lambda/c$. For a massless particle, the single-particle Lagrangian is therefore simply given by the last term in Eq. (2.8), i.e.,

$$L(v;\lambda)\equiv -\frac{\lambda}{2}v_\alpha v^\alpha. \tag{2.11}$$

This Lagrangian appears in the bosonic part of the Lagrangian for a spinning particle [38]. The Lagrange multiplier λ^{-1} corresponds to the "einbein" which describes the square-root metric $e=\sqrt{g}$ along the world line in a particular gauge where the world line is parametrized by time.

B. Lagrangian density for a free ideal fluid

We now discuss the passage from the finite-dimensional single-particle Lagrangian formalism based on Eq. (2.1) to an infinite-dimensional fluid Lagrangian formalism. To obtain a Lagrangian density for a fluid composed of such particles, we multiply Eq. (2.1) by the reference-frame density n, noting that the proper density is $N \equiv n\gamma^{-1}$. The Lagrangian for a cold ideal fluid is therefore

$$\mathcal{L}_0 = -mc^2 N = -mc^2 n\sqrt{v^\alpha v_\alpha} = -mc^2\sqrt{J^\alpha J_\alpha}, \tag{2.12}$$

where $J^\alpha = nv^\alpha = \langle \bar{\psi}\hat{\gamma}^\alpha \psi/c\rangle$ is the flux four-vector with a suitable ensemble average $\langle \cdots \rangle$. The Lorentz invariance is manifest in the last expression.

Another contribution to the Lagrangian density of an ideal fluid is the term $-N\epsilon(N,S)$ associated with the internal energy density of the fluid in its rest frame, where the internal energy $\epsilon(N,S)$ is a function of the proper fluid density N and its entropy S (a Lorentz scalar). By combining these two terms, the Lagrangian density for a free relativistic fluid is therefore written as

$$\mathcal{L}_0 = -N[mc^2 + \epsilon(N,S)] \equiv -N\,\varepsilon(N,S), \tag{2.13}$$

where the total internal energy

$$\varepsilon(N,S) \equiv mc^2 + \epsilon(N,S) \tag{2.14}$$

includes the particle's rest energy.

As discussed above, the single-particle Lagrangian for a free massless particle is given by Eq. (2.11). The Lagrangian density for a cold ideal fluid composed of massless neutrinos is therefore given as

$$\mathcal{L}_0 \equiv -\frac{\lambda'_\nu}{2} J_\nu \cdot J_\nu = -\frac{n_\nu \lambda_\nu}{2} v_\nu \cdot v_\nu, \tag{2.15}$$

where λ'_ν is a Lorentz-scalar Lagrange multiplier field. The last expression is equivalent upon changing the variable $\lambda_\nu = n_\nu \lambda'_\nu$.

III. CONSTRAINED VARIATIONAL PRINCIPLE

The self-consistent nonlinear neutrino-plasma fluid equations presented in this paper are derived from the variational principle

$$\delta \int d^4x\ \mathcal{L}(A^\alpha, F^{\alpha\beta}; N_s, u_s^\alpha, S_s) = 0, \tag{3.1}$$

where, in addition to its dependence on the electromagnetic four-potential A^α and the Faraday tensor $F^{\alpha\beta}$, the Lagrangian density $\mathcal{L}$ depends on the proper density $N_s \equiv n_s\gamma_s^{-1}$, the normalized fluid four-velocity $u_s^\alpha \equiv (\gamma_s, \mathbf{u}_s)$, and the proper internal energy (per particle) ε_s for each fluid species s (here, $s=\sigma$ denotes a plasma-fluid species and $s=\nu$ denotes a neutrino-fluid species).

The proper internal energy $\varepsilon_s(N_s, S_s)$ includes the particle's rest energy [see Eq. (2.13)] and depends on the proper density N_s and the entropy S_s (a Lorentz scalar). The first law of thermodynamics [39–41] is written as

$$d\varepsilon_s = T_s\, dS_s - p_s\, dN_s^{-1}, \tag{3.2}$$

where T_s is the proper temperature and p_s is the scalar pressure for fluid species s. In what follows we use the chemical potential for each fluid species s:

$$\mu_s \equiv \partial(\varepsilon_s N_s)/\partial N_s = \varepsilon_s + p_s/N_s, \tag{3.3}$$

which represents the total energy required to create a particle of species s and inject it into a fluid sample composed of particles of the same species. Associated with the definition for the chemical potential (3.3), we also use the identity

$$\partial^\alpha \mu_s = T_s\, \partial^\alpha S_s + N_s^{-1}\, \partial^\alpha p_s. \tag{3.4}$$

Note that the independent fluid variables for each fluid species are N_s, u_s^α, and S_s although other combinations are possible [32].

The Lagrangian formulation for the nonlinear interaction between neutrino and plasma fluids in the presence of an electromagnetic field is expressed in terms of the Lagrangian density

$$\mathcal{L} = -\sum_{s=\sigma,\nu} N_s\,\varepsilon_s - \sum_\sigma J_\sigma \cdot \left(q_\sigma A + \sum_\nu G_{\sigma\nu} J_\nu \right) + \frac{1}{16\pi} \mathsf{F}:\mathsf{F}, \tag{3.5}$$

where $\mathsf{F}:\mathsf{F} \equiv F^{\alpha\beta}F_{\beta\alpha}$. The first term in Eq. (3.5) denotes the total internal energy density of fluid s. The second term denotes the standard coupling between a charged (plasma) fluid and an electromagnetic field. The third term denotes the coupling between the neutrino-fluid species ν and the plasma-fluid species σ. Note that the second and third terms can be written as $\Sigma_\sigma n_\sigma V_\sigma$, where the single-particle velocity $\mathbf{v}$ in Eq. (1.8) is replaced with the fluid velocity $\mathbf{v}_\sigma$. The fourth term is the familiar electromagnetic field Lagrangian.

In the variational principle (3.1), the variation $\delta\mathcal{L}$ is explicitly written as

$$\delta\mathcal{L} \equiv \delta A \cdot \frac{\partial \mathcal{L}}{\partial A} - \delta\mathsf{F}:\frac{\partial \mathcal{L}}{\partial \mathsf{F}} + \sum_s \left(\delta N_s \frac{\partial \mathcal{L}}{\partial N_s} + \delta u_s \cdot \frac{\partial \mathcal{L}}{\partial u_s} + \delta S_s \frac{\partial \mathcal{L}}{\partial S_s} \right), \tag{3.6}$$

where $\partial F_{\alpha\beta} = \partial_\alpha \delta A_\alpha - \partial_\alpha \delta A_\alpha$ so that the second term in Eq. (3.6) can also be written as $+2\partial\delta A:\partial\mathcal{L}/\partial\mathsf{F}$. In constrast to other variational principles [32,42], the Eulerian variations δN_s, δu_s and δS_s in Eq. (3.6) are not arbitrary but are instead *constrained*.

To obtain the correct Eulerian variation, recall that the variation of the fluid motion is an infinitesimal displacement of the fluid elements. With a fluid element s described by the four-coordinate x_s^α, the normalized velocity four-vector is given by

$$u_s^\alpha(x) = \frac{dx_s^\alpha}{d\tau}\left(\frac{dx_s}{d\tau} \cdot \frac{dx_s}{d\tau} \right)^{-1/2} \equiv \left| \frac{dx_s}{d\tau} \right|^{-1} \frac{dx_s^\alpha}{d\tau}, \tag{3.7}$$

where τ parametrizes the world line of the fluid element. Under the infinitesimal displacement $x_s^\alpha \rightarrow x_s^\alpha + \delta\xi_s^\alpha$ [with $\delta\xi_s^\alpha \equiv (\delta\xi_s^0, \delta\boldsymbol{\xi}_s)$], the apparent variation at a position following a fluid element along its world line is

$$\begin{aligned} \frac{d\,\delta\xi_s^\alpha}{d\tau}\left|\frac{dx_s}{d\tau}\right|^{-1} - \frac{dx_s^\alpha}{d\tau}\left|\frac{dx_s}{d\tau}\right|^{-3} \frac{d\,\delta\xi_s}{d\tau}\cdot\frac{dx_s}{d\tau} \\ = (u_s\cdot\partial)\,\delta\xi_s^\alpha - u_s^\alpha\,[u_{s\beta}(u_s\cdot\partial)\,\delta\xi_s^\beta] \\ \equiv h_s^{\alpha\beta}(u_s\cdot\partial)\,\delta\xi_{s\beta}, \end{aligned} \tag{3.8}$$

where $u_s\cdot\partial \equiv |dx_s/d\tau|^{-1}\; d/d\tau$ and

$$h_s^{\alpha\beta} \equiv g^{\alpha\beta} - u_s^\alpha u_s^\beta \tag{3.9}$$

is a symmetric projection tensor [40] (i.e., $\mathsf{h}_s\cdot u_s \equiv 0$). The Eulerian variation at a fixed space-time location is therefore given by [33]

$$\delta u_s^\alpha(x) = h_s^{\alpha\beta}(u_s\cdot\partial)\,\delta\xi_{s\beta} - (\delta\xi_s\cdot\partial)u_s^\alpha. \tag{3.10}$$

It is easy to check that this variation preserves $u^\alpha u_\alpha = 1$.

The variation of the proper density N_s can be obtained by the requirement that the quantity

$$N_s\left(\frac{dx_s}{d\tau}\cdot\frac{dx_s}{d\tau}\right)^{-1/2} d^4x \tag{3.11}$$

should be kept intact (i.e., mass is conserved). The factor in parentheses is the induced metric along the world line. This requirement fixes the variation at a position following a fluid element along its world line as $-N_s[(\partial\cdot\delta\xi_s) - u_{s\beta}(u_s\cdot\partial)\,\delta\xi_s^\beta] = -N_s\,[h_s^{\alpha\beta}\partial_\alpha\delta\xi_{s\beta}]$, and hence the Eulerian variation is given by

$$\delta N_s = -(\delta\xi_s\cdot\partial)N_s - N_s\;\mathsf{h}_s:\partial\,\delta\xi_s. \tag{3.12}$$

It is straightforward to check that the above variations Eqs. (3.10) and (3.12) are consistent with the conservation law

$$\partial_\alpha J_s^\alpha = 0 \tag{3.13}$$

of the flux four-vector $J_s^\alpha = N_s u_s^\alpha$. It is useful to know its variation, which can be easily calculated using Eqs. (3.10) and (3.12):

$$\delta J_s^\alpha = \partial_\beta(J_s^\beta\,\delta\xi_s^\alpha - J_s^\alpha\,\delta\xi_s^\beta), \tag{3.14}$$

where the conservation law (3.13) has been used.

Finally, the nondissipative flow conserves entropy along the world line,

$$(u_s\cdot\partial)S_s = 0. \tag{3.15}$$

To be consistent with the variation Eq. (3.10), we find

$$\delta S_s = -(\delta\xi_s\cdot\partial)S_s. \tag{3.16}$$

The expressions (3.10), (3.12), and (3.16) give the correct relativistic generalizations of the (nonrelativistic) constrained Eulerian variations [43]; see the Appendix for a geometric interpretation of Eqs. (3.10), (3.12), (3.14), and (3.16). An alternative variational principle would introduce $\partial\cdot J_s = 0 = u_s\cdot\partial S_s$ explicitly in the Lagrangian density by means of Lagrange multipliers [32].

IV. SELF-CONSISTENT NONLINEAR NEUTRINO-PLASMA FLUID EQUATIONS

We now proceed with the variational derivation of the dynamical equations for self-consistent neutrino-plasma fluid interactions. In deriving these equations, we use the thermodynamic relations (3.2)–(3.4) as well as the continuity and entropy equations (3.13) and (3.15) for each fluid species s.

By rearranging terms in the variational equation (3.6) so as to isolate the variation four-vectors $\delta\xi_s$ and δA, we find

$$\begin{aligned} \delta\mathcal{L} \equiv \partial\cdot\mathcal{J} - \sum_s\;\delta\xi_s\cdot\left[\partial_s\mathcal{L} + \partial\cdot\left(u_s\,\frac{\partial\mathcal{L}}{\partial u_s}\cdot\mathsf{h}_s - N_s\frac{\partial\mathcal{L}}{\partial N_s}\,\mathsf{h}_s\right)\right] \\ + \delta A\cdot\left(\frac{\partial\mathcal{L}}{\partial A} - 2\,\partial\cdot\frac{\partial\mathcal{L}}{\partial\mathsf{F}}\right), \end{aligned} \tag{4.1}$$

where $\partial_s\mathcal{L} \equiv \partial N_s\,(\partial\mathcal{L}/\partial N_s) + \partial u_s\cdot(\partial\mathcal{L}/\partial u_s) + \partial S_s\,(\partial\mathcal{L}/\partial S_s)$, and the Noether four-density $\mathcal{J}$ is expressed in terms of $\delta\xi_s$ and δA as

$$\mathcal{J} \equiv \sum_s\;\left(u_s\,\frac{\partial\mathcal{L}}{\partial u_s}\cdot\mathsf{h}_s - N_s\frac{\partial\mathcal{L}}{\partial N_s}\,\mathsf{h}_s\right)\cdot\delta\xi_s + 2\,\frac{\partial\mathcal{L}}{\partial\mathsf{F}}\cdot\delta A. \tag{4.2}$$

When performing the variational principle (3.1), with $\delta\mathcal{L}$ given by Eq. (4.1), we consider only variations $\delta\xi_s$ and δA that vanish on the integration boundary. Hence, the Noether density $\mathcal{J}$ in Eq. (4.1) does not contribute to the dynamical equations.

A. Plasma-fluid momentum equation

First, we derive the relativistic plasma-fluid four-momentum equation. Upon variation with respect to $\delta\xi_\sigma$ in Eq. (3.1), we obtain

$$\begin{aligned} 0 = \left(\partial N_\sigma\,\frac{\partial\mathcal{L}}{\partial N_\sigma} + \partial u_\sigma\cdot\frac{\partial\mathcal{L}}{\partial u_\sigma} + \partial S_\sigma\,\frac{\partial\mathcal{L}}{\partial S_\sigma}\right) \\ + \partial\cdot\left(u_\sigma\,\frac{\partial\mathcal{L}}{\partial u_\sigma}\cdot\mathsf{h}_\sigma - N_\sigma\frac{\partial\mathcal{L}}{\partial N_\sigma}\,\mathsf{h}_\sigma\right). \end{aligned} \tag{4.3}$$

By substituting appropriate derivatives of the Lagrangian density $\mathcal{L}$ and using the constraint equations (3.13) and (3.15) and the thermodynamic relations (3.2)–(3.4), this equation becomes the relativistic plasma-fluid four-momentum (covariant) equation

$$u_\sigma\cdot\partial(\mu_\sigma\,u_\sigma) = N_\sigma^{-1}\partial p_\sigma + \left(q_\sigma\,\mathsf{F} + \sum_\nu\;G_{\sigma\nu}\,\mathsf{M}_\nu\right)\cdot u_\sigma, \tag{4.4}$$

where

$$M_\nu^{\alpha\beta} \equiv \partial^\alpha J_\nu^\beta - \partial^\beta J_\nu^\alpha \tag{4.5}$$

is an antisymmetric tensor that represents the influence of the neutrino background medium [44]. This tensor satisfies the

Maxwell-like equation $\partial^\rho M_\nu^{\alpha\beta}+\partial^\alpha M_\nu^{\beta\rho}+\partial^\beta M_\nu^{\rho\alpha}\equiv 0$ and its divergence is $\partial_\alpha M_\nu^{\alpha\beta}\equiv\Box J_\nu^\beta$, where $\Box\equiv\partial\cdot\partial$ and the continuity equation $\partial\cdot J_\nu=0$ for the neutrino fluid was used.

Separating the space and time components in Eq. (4.4) (i.e., using the $3+1$ notation), the spatial components of the plasma-fluid four-momentum equation (4.4) yield

$$(\partial_t+\mathbf{v}_\sigma\cdot\nabla)(\mu_\sigma\gamma_\sigma\mathbf{v}_\sigma/c^2)=-n_\sigma^{-1}\nabla p_\sigma+q_\sigma\left(\mathbf{E}+\frac{\mathbf{v}_\sigma}{c}\times\mathbf{B}\right)+\mathbf{f}_\sigma, \quad (4.6)$$

where $\mathbf{f}_\sigma$ is the neutrino-induced ponderomotive force (averaged over neutrino species) on the plasma-fluid species σ, defined as

$$\mathbf{f}_\sigma\equiv\sum_\nu G_{\sigma\nu}\left[-\left(\nabla n_\nu+\frac{1}{c}\frac{\partial\boldsymbol{J}_\nu}{\partial t}\right)+\frac{\mathbf{v}_\sigma}{c}\times\nabla\times\boldsymbol{J}_\nu\right]. \quad (4.7)$$

The neutrino-induced ponderomotive force $\mathbf{f}_\sigma$ is composed of three terms: an electrostaticlike term ∇n_ν, an inductivelike term $\partial_t\boldsymbol{J}_\nu$, and a magneticlike term $\nabla\times\boldsymbol{J}_\nu$. This terminology is obviously motivated by the similarities with the electromagnetic force on a charged particle. In previous work by Silva *et al.* [17], only the electrostaticlike term was retained in the neutrino-induced ponderomotive force, i.e., the neutrino particle flux $\boldsymbol{J}_\nu$ was discarded under the assumption of isotropic neutrino and plasma fluids.

B. Neutrino-fluid momentum equation

Next, we derive the relativistic neutrino-fluid four-momentum equation; the limiting case of zero neutrino masses is treated below Eq. (4.12). Upon variation with respect to $\delta\xi_\nu$ in Eq. (3.1), we obtain

$$0=\left(\partial N_\nu\frac{\partial\mathcal{L}}{\partial N_\nu}+\partial u_\nu\cdot\frac{\partial\mathcal{L}}{\partial u_\nu}+\partial S_\nu\frac{\partial\mathcal{L}}{\partial S_\nu}\right)+\partial\cdot\left(u_\nu\frac{\partial\mathcal{L}}{\partial u_\nu}\cdot\mathsf{h}_\nu-N_\nu\frac{\partial\mathcal{L}}{\partial N_\nu}\mathsf{h}_\nu\right). \quad (4.8)$$

On substituting derivatives of $\mathcal{L}$ and using the thermodynamic relations (3.2)–(3.4), this equation becomes the relativistic neutrino-fluid four-momentum equation

$$u_\nu\cdot\partial(\mu_\nu u_\nu)=N_\nu^{-1}\partial p_\nu+\sum_\sigma G_{\sigma\nu}\mathsf{M}_\sigma\cdot u_\nu, \quad (4.9)$$

where

$$M_\sigma^{\alpha\beta}\equiv\partial^\alpha J_\sigma^\beta-\partial^\beta J_\sigma^\alpha \quad (4.10)$$

is another antisymmetric tensor which represents the influence of the background medium. This tensor satisfies the Maxwell-like equation $\partial^\rho M_\sigma^{\alpha\beta}+\partial^\alpha M_\sigma^{\beta\rho}+\partial^\beta M_\sigma^{\rho\alpha}\equiv 0$ and its divergence is $\partial_\alpha M_\sigma^{\alpha\beta}\equiv\Box J_\sigma^\beta$, where the continuity equation $\partial J_\sigma=0$ for the plasma fluid was used. In Eq. (4.9), we note that the neutrino fluid is thus under the influence of an electromagneticlike force induced by nonuniform plasma flows. We also note that the symmetry between the ponderomotive forces (4.5) and (4.10) is a result of the symmetry of the neutrino-plasma interaction term $(\Sigma_\sigma\Sigma_\nu G_{\sigma\nu}J_\sigma\cdot J_\nu)$ in the Lagrangian density (3.5).

Using the $3+1$ notation, the spatial components of neutrino-fluid four-momentum equation (4.9) yield

$$(\partial_t+\mathbf{v}_\nu\cdot\nabla)(\mu_\nu\gamma_\nu\mathbf{v}_\nu/c^2)=-n_\nu^{-1}\nabla p_\nu+\mathbf{f}_\nu, \quad (4.11)$$

where $\mathbf{f}_\nu$ is the plasma-induced ponderomotive force (averaged over plasma-particle species) on the neutrino-fluid species ν, defined as

$$\mathbf{f}_\nu\equiv\sum_\sigma G_{\sigma\nu}\left[-\left(\nabla n_\sigma+\frac{1}{c}\frac{\partial\boldsymbol{J}_\sigma}{\partial t}\right)+\frac{\mathbf{v}_\nu}{c}\times\nabla\times\boldsymbol{J}_\sigma\right]. \quad (4.12)$$

The plasma-induced ponderomotive force $\mathbf{f}_\nu$ on the neutrino fluid is composed of three terms: an electrostaticlike term ∇n_σ, an inductivelike term $\partial_t\boldsymbol{J}_\sigma$, and a magneticlike term $\nabla\times\boldsymbol{J}_\sigma$.

We now discuss the case of a cold ideal fluid composed of massless neutrinos. Variation of the neutrino part of the Lagrangian density

$$\mathcal{L}_\nu\equiv-\tfrac{1}{2}n_\nu\lambda_\nu v_\nu\cdot v_\nu-\sum_\sigma G_{\sigma\nu}J_\sigma\cdot J_\nu$$

with respect to $\delta\xi_\nu$ yields

$$\delta\mathcal{L}_\nu\equiv-n_\nu\lambda_\nu\delta v_\nu\cdot v_\nu-\sum_\sigma G_{\sigma\nu}J_\sigma\cdot\delta J_\nu, \quad (4.13)$$

where we used the constraint $v_\nu\cdot v_\nu\equiv 0$. Using $\delta J_\nu\equiv\partial\cdot(J_\nu\delta\xi_\nu-\delta\xi_\nu J_\nu)$ and $\delta v_\nu\cdot J_\nu=\partial\cdot(J_\nu\delta\xi_\nu)\cdot v_\nu$, the variation equation (4.13) becomes

$$\delta\mathcal{L}_\nu\equiv\partial\cdot\mathcal{J}+n_\nu\delta\xi_\nu\cdot\left(v_\nu\cdot\partial(\lambda_\nu v_\nu)-\sum_\sigma G_{\sigma\nu}\mathsf{M}_\sigma\cdot v_\nu\right), \quad (4.14)$$

where the tensor M_σ is defined in Eq. (4.10) and the Noether density is

$$\mathcal{J}\equiv\delta\xi_\nu\cdot\left[\mathsf{g}\,G_{\sigma\nu}J_\nu\cdot J_\sigma-J_\nu\left(\lambda_\nu v_\nu+\sum_\sigma G_{\sigma\nu}J_\sigma\right)\right]. \quad (4.15)$$

From Eq. (4.14) the variational principle $\int\delta\mathcal{L}_\nu d^4x=0$ yields the cold neutrino fluid equation

$$v_\nu\cdot\partial(\lambda_\nu v_\nu)=\sum_\sigma G_{\sigma\nu}\mathsf{M}_\sigma\cdot v_\nu. \quad (4.16)$$

In the cold-fluid limit, on the other hand, Eq. (4.9) yields $u_\nu\cdot\partial(\gamma_\nu^{-1}\lambda_\nu u_\nu)=\Sigma_\sigma G_{\sigma\nu}\mathsf{M}_\sigma\cdot u_\nu$, where λ_ν is the neutrino energy. By substituting $u_\nu\equiv\gamma_\nu v_\nu$ into this expression, we readily check that Eqs. (4.9) and (4.16) are identical in the massless-neutrino cold-fluid limit and that Eq. (4.9) can in fact be used to describe neutrino-fluid dynamics with arbitrary neutrino mass.

C. Maxwell equations

The remaining equations are obtained from the variational principle (3.1) upon variations with respect to the four-potential δA^α. One thus obtains

$$0=\frac{\partial \mathcal{L}}{\partial A}-2\,\partial\cdot\frac{\partial \mathcal{L}}{\partial \mathsf{F}}. \tag{4.17}$$

With substitution of derivatives of $\mathcal{L}$, this equation becomes the Maxwell equation

$$\partial\cdot\mathsf{F}=4\pi\sum_\sigma \; q_\sigma J_\sigma . \tag{4.18}$$

Using the $3+1$ notation, we recover one-half of the familiar Maxwell equations from Eq. (4.18). The other half is expressed in terms of the Faraday tensor alone as

$$\partial^\rho F^{\alpha\beta}+\partial^\alpha F^{\beta\rho}+\partial^\beta F^{\rho\alpha}\equiv 0, \tag{4.19}$$

which, using the $3+1$ notation, yields $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{E}+c^{-1}\partial_t\mathbf{B}=0$.

D. Energy-momentum conservation laws

Since the dynamical equations (4.3), (4.8), and (4.17) are true for arbitrary variations $(\delta\xi_\sigma,\delta\xi_\nu)$ and δA (subject to boundary conditions), the variational equation (4.1) becomes

$$\delta\mathcal{L}\equiv\partial\cdot\mathcal{J}, \tag{4.20}$$

which we henceforth refer to as the Noether equation. We now discuss Noether symmetries of the Lagrangian density (3.5) based on the Noether equation (4.20).

For this purpose, we consider infinitesimal translations $x^\alpha\rightarrow x^\alpha+\delta x^\alpha$ generated by the infinitesimal displacement four-vector field δx. Under this transformation, the Lagrangian density $\mathcal{L}$ changes by

$$\delta\mathcal{L}\equiv-\partial\cdot(\delta x\;\mathcal{L}). \tag{4.21}$$

Next, we introduce the following explicit expressions for $(\delta\xi_\sigma,\delta\xi_\nu)$ and δA in terms of the infinitesimal generating four-vector δx:

$$\begin{aligned}\delta\xi_s&\equiv\mathsf{h}_s\cdot\delta x,\\ \delta A&\equiv\mathsf{F}\cdot\delta x-\partial(A\cdot\delta x),\end{aligned} \tag{4.22}$$

where the symmetric tensor h_s is defined in Eq. (3.9). (These expressions are given geometric interpretations in the Appendix.)

Substituting Eq. (4.22) in the Noether density (4.2), we find

$$\begin{aligned}\mathcal{J}=&\left[2\,\frac{\partial\mathcal{L}}{\partial\mathsf{F}}\cdot\mathsf{F}+\sum_s\;\left(u_s\,\frac{\partial\mathcal{L}}{\partial u_s}\cdot\mathsf{h}_s-N_s\frac{\partial\mathcal{L}}{\partial N_s}\,\mathsf{h}_s\right)\right]\cdot\delta x\\ &+2\,\partial(A\cdot\delta x)\cdot\frac{\partial\mathcal{L}}{\partial\mathsf{F}},\end{aligned} \tag{4.23}$$

where we have used the identity $\mathsf{h}_s\cdot\mathsf{h}_s=\mathsf{h}_s$ in writing the second and third terms. Making use of the Maxwell equation (4.18), the last term in Eq. (4.23) can be rearranged as

$$2\,\partial(A\cdot\delta x)\cdot\frac{\partial\mathcal{L}}{\partial\mathsf{F}}=\partial\cdot\left(2\,(A\cdot\delta x)\,\frac{\partial\mathcal{L}}{\partial\mathsf{F}}\right)-(A\cdot\delta x)\,\frac{\partial\mathcal{L}}{\partial A}. \tag{4.24}$$

We now note that the expression for $\partial\cdot\mathcal{J}$ in Eq. (4.20) is invariant under the transformation $\mathcal{J}\rightarrow\mathcal{J}+\partial\cdot\mathsf{K}$, where K is an antisymmetric tensor (for which $\partial^2_{\alpha\beta}K^{\alpha\beta}\equiv 0$) which vanishes on the integration boundary in Eq. (3.1). Since the first term on the right side of Eq. (4.24) is such a term, it can be transformed away, and the final expression for the Noether density is therefore

$$\mathcal{J}=\left[2\,\frac{\partial\mathcal{L}}{\partial\mathsf{F}}\cdot\mathsf{F}-\frac{\partial\mathcal{L}}{\partial A}\,A+\sum_s\;\left(u_s\,\frac{\partial\mathcal{L}}{\partial u_s}\cdot\mathsf{h}_s-N_s\frac{\partial\mathcal{L}}{\partial N_s}\,\mathsf{h}_s\right)\right]\cdot\delta x. \tag{4.25}$$

Substituting Eq. (4.21) into Eq. (4.20), the Noether equation becomes $\partial\cdot(\mathcal{J}+\delta x\,\mathcal{L})=0$. We define the symmetric energy-momentum tensor T from the expression

$$\mathcal{J}+\delta x\;\mathcal{L}\equiv-\,\mathsf{T}\cdot\delta x, \tag{4.26}$$

where, using Eq. (4.25), the energy-momentum tensor T is explicitly given as

$$\begin{aligned}\mathsf{T}=&-\,\mathsf{g}\;\mathcal{L}-\left(2\,\frac{\partial\mathcal{L}}{\partial\mathsf{F}}\cdot\mathsf{F}-\frac{\partial\mathcal{L}}{\partial A}\,A\right)\\ &-\sum_s\;\left(u_s\,\frac{\partial\mathcal{L}}{\partial u_s}\cdot\mathsf{h}_s-N_s\frac{\partial\mathcal{L}}{\partial N_s}\,\mathsf{h}_s\right).\end{aligned} \tag{4.27}$$

For a constant translation δx, the Noether equation (4.20) then becomes

$$0=\partial\cdot\mathsf{T}, \tag{4.28}$$

where using the Lagrangian density (3.5) and its derivatives in Eq. (4.27), we find

$$\begin{aligned}T^{\alpha\beta}=&\frac{1}{4\pi}\left(F^\alpha_{\;\kappa}F^{\kappa\beta}-\frac{g^{\alpha\beta}}{4}\,\mathsf{F}{:}\mathsf{F}\right)+\sum_s\;(N_s\mu_s\;u_s^\alpha u_s^\beta-p_s\,g^{\alpha\beta})\\ &+\sum_\sigma\;\sum_\nu\;\;G_{\sigma\nu}\,(J^\alpha_\sigma J^\beta_\nu+J^\alpha_\nu J^\beta_\sigma-g^{\alpha\beta}\;J_\sigma\cdot J_\nu).\end{aligned} \tag{4.29}$$

This energy-momentum tensor contains the usual terms associated with an electromagnetic field and a free relativistic ideal fluid [39–41]. It also contains the energy-momentum terms associated with collective neutrino-plasma interactions (third set of terms).

The energy-momentum transfer between the electromagnetic-plasma background and the neutrinos can now be investigated. Such a process is relevant to supernova explosions, for example, where approximately 1% of the neutrino energy needs to be transferred to the surrounding plasma. First, we define the electromagnetic-plasma (EMP) energy-momentum tensor:

$$\mathsf{T}_{\mathrm{EMP}} \equiv \frac{1}{4\pi}\left(\mathsf{F}\cdot\mathsf{F} - \frac{\mathsf{g}}{4}\,\mathsf{F}:\mathsf{F}\right) + \sum_{\sigma} \left(N_\sigma \mu_\sigma\, u_\sigma u_\sigma - p_\sigma\, \mathsf{g}\right)$$
$$\equiv \mathsf{T}_{\mathrm{EM}} + \mathsf{T}_{\mathrm{P}}, \tag{4.30}$$

and, using the exact energy-momentum conservation law (4.28) as well as the dynamical equations (4.4), (4.9), and (4.18), we find

$$\partial\cdot\mathsf{T}_{\mathrm{EMP}} = \sum_{\sigma}\left(\sum_{\nu} G_{\sigma\nu}\,\mathsf{M}_\nu\right)\cdot J_\sigma. \tag{4.31}$$

This equation describes how energy and momentum are transferred from the neutrinos to the electromagnetic field and the background plasma. Note how the transfer of energy-momentum between an electromagnetic plasma and neutrinos is very much like the transfer of energy between a plasma (P) and an electromagnetic field (i.e., $\partial\cdot\mathsf{T}_{\mathrm{P}} = \Sigma_\sigma\, q_\sigma \mathsf{F}\cdot J_\sigma$) in the absence of neutrinos.

We note that in addition to energy and momentum, wave action [15] can be transferred between the neutrinos and the electromagnetic-plasma background. In this case, electromagnetic waves and/or plasma waves can be excited by resonant three-wave processes.

V. MAGNETIC FIELD GENERATION AND HELICITY PRODUCTION BY COLLECTIVE NEUTRINO-PLASMA INTERACTIONS

An important application of the process of energy-momentum transfer associated with collective electromagnetic-plasma–neutrino interactions is the possibility of generating magnetic fields in an unmagnetized plasma as a result of collective neutrino-plasma interactions. Such process might be relevant to the problem of magnetogenesis and the production of magnetic helicity in the early universe [45–47]. A similar process of magnetic field generation has been observed in laser-plasma interactions [26–29].

According to our neutrino-plasma fluid model [based on Eqs. (4.4), (4.9), and (4.18)], the strength of the magnetic field generated by neutrino-plasma interactions scales as the first power in the Fermi weak-interaction constant G_F. In what follows, we thus refer to magnetic fields generated by classical plasma processes (e.g., the Biermann battery effect and the nonlinear dynamo effect) as zeroth-order fields while those generated by collective neutrino-plasma interactions are first-order fields. Second-order fields, for example, might be produced by processes such as $\sigma'\rightarrow\nu\rightarrow\sigma\rightarrow$EM, where the first plasma-particle species (σ') need not be charged (e.g., neutrons). In this section, we investigate the role played by collective neutrino-plasma interactions in generating magnetic fields and magnetic helicity as well as magnetic equilibrium.

A. Magnetic field generation

An equation describing magnetic field generation resulting from collective neutrino-plasma interactions is derived as follows. We begin with Faraday's law

$$\frac{\partial\mathbf{B}}{\partial t} = -c\,\nabla\times\mathbf{E}, \tag{5.1}$$

where for a given plasma-particle species σ [using Eq. (4.7)], the electric field $\mathbf{E}$ is expressed as

$$\mathbf{E} \equiv \frac{1}{q_\sigma}(\mathbf{F}_\sigma - \mathbf{f}_\sigma) - \frac{\mathbf{v}_\sigma}{c}\times\mathbf{B}, \tag{5.2}$$

where $\mathbf{f}_\sigma$ is the neutrino-induced ponderomotive force given by Eq. (4.7) and

$$\mathbf{F}_\sigma \equiv \frac{\partial\mathbf{P}_\sigma}{\partial t} + \mathbf{v}_\sigma\cdot\nabla\mathbf{P}_\sigma + n_\sigma^{-1}\,\nabla p_\sigma, \tag{5.3}$$

with $\mathbf{P}_\sigma \equiv (\mu_\sigma/c^2)\gamma_\sigma\mathbf{v}_\sigma$ the generalized momentum for plasma-fluid species σ.

Since the electric field $\mathbf{E}$ is common to all charged-particle species, we multiply Eq. (5.2) on both sides by q_σ^2 and sum over all charged-particle species present in the plasma. Defining $\Sigma_\sigma q_\sigma^2 \equiv Q^2$, the electric field $\mathbf{E}$ is then given as

$$\mathbf{E} = \sum_\sigma \frac{q_\sigma}{Q^2}(\mathbf{F}_\sigma - \mathbf{f}_\sigma) - \left(\sum_\sigma \frac{q_\sigma^2\mathbf{v}_\sigma}{cQ^2}\right)\times\mathbf{B}. \tag{5.4}$$

Substituting explicit expressions for $\mathbf{F}_\sigma$ and $\mathbf{f}_\sigma$, we obtain

$$\mathbf{E} \equiv \sum_\sigma \frac{q_\sigma}{Q^2}[\partial_t\mathbf{\Pi}_\sigma - \mathbf{v}_\sigma\times\nabla\times\mathbf{\Pi}_\sigma + \nabla\chi_\sigma + S_\sigma\,\nabla(\gamma_\sigma^{-1}T_\sigma)]$$
$$-\left(\sum_\sigma \frac{q_\sigma^2\mathbf{v}_\sigma}{cQ^2}\right)\times\mathbf{B}, \tag{5.5}$$

where $\gamma_\sigma^{-1}T_\sigma$ is the temperature in the laboratory reference frame and

$$\mathbf{\Pi}_\sigma \equiv \mathbf{P}_\sigma + \sum_\nu G_{\sigma\nu}\mathbf{J}_\nu/c,$$
$$\chi_\sigma \equiv \sum_\nu G_{\sigma\nu}\,n_\nu + \gamma_\sigma\,\mu_\sigma - \gamma_\sigma^{-1}T_\sigma S_\sigma. \tag{5.6}$$

Equation (5.5) can then be substituted for the electric field into Faraday's law (5.1) to give

$$\frac{\partial\mathbf{B}}{\partial t} = \sum_\sigma \frac{c\,q_\sigma}{Q^2}[\nabla(\gamma_\sigma^{-1}T_\sigma)\times\nabla S_\sigma$$
$$-\nabla\times(\partial_t\mathbf{P}_\sigma - \mathbf{v}_\sigma\times\nabla\times\mathbf{P}_\sigma)]$$
$$+\sum_\sigma \frac{q_\sigma^2}{Q^2}\,\nabla\times(\mathbf{v}_\sigma\times\mathbf{B})$$
$$-\sum_\nu\sum_\sigma \frac{q_\sigma G_{\sigma\nu}}{Q^2}\,\nabla\times(\partial_t\mathbf{J}_\nu - \mathbf{v}_\sigma\times\nabla\times\mathbf{J}_\nu). \tag{5.7}$$

The first collection of terms (linear in q_σ) on the right side of Eq. (5.7) includes the so-called Biermann battery term ($\nabla n_\sigma^{-1}\times\nabla T_\sigma$) [27,28,48] while the second term (proportional to q_σ^2) represents the nonlinear dynamo effect. These classical (zeroth-order) terms have been known to play important roles in the generation of magnetic fields during laser-plasma interactions [26–29] as well as the evolution of cosmic and galactic magnetic fields [48].

The last collection of terms (proportional to $q_\sigma G_{\sigma\nu}$) in Eq. (5.7) is associated with collective neutrino-plasma interactions. Here, the neutrino-flux vorticity ($\nabla\times\boldsymbol{J}_\nu$) plays a fundamental role in generating first-order magnetic fields; such terms are completely missing from previous works [30,31].

According to Eq. (5.7), the electrostatic part of the neutrino-induced ponderomotive force Eq. (4.7) does not play any role in generating magnetic fields. Indeed, for each neutrino-fluid species ν, we have $\nabla\times[(\Sigma_s q_s G_{s\nu})\nabla n_\nu]=0$, independent of the plasma-fluid composition. The neutrino-induced ponderomotive force on plasma particles of species σ actually given in [14,30,31] is $-n_\sigma^{-1}(\Sigma_{s'} G_{s'\nu} n_{s'})\nabla n_\nu \equiv \mathbf{f}_\sigma^{(B)}$; this expression improperly involves a sum of plasma-particle species ($\Sigma_{s'}$) instead of the sum over neutrino species (Σ_ν) as it appears in Eq. (4.7). Shukla *et al.* [31] then go on to develop a model for magnetic field generation based on the fact that $\nabla\times\mathbf{f}_\sigma^{(B)}\neq 0$ for a plasma with multiple particle species. Since the sum over plasma-particle species ($\Sigma_{s'}$) appearing in $\mathbf{f}_\sigma^{(B)}$ is inappropriate, however, the conclusion drawn by Shukla *et al.* [31] that magnetic fields can be generated in a plasma composed of neutrons ($\sigma=n$) and electrons ($\sigma=e$) by terms such as $\nabla(n_n/n_e)\times\nabla n_\nu$ is incorrect [49].

For a primordial plasma, we note that the Biermann-battery term could be small unless the terms $\nabla(\gamma_\sigma^{-1}T_\sigma)\times\nabla S_\sigma$ and $\nabla(\gamma_{\bar\sigma}^{-1}T_{\bar\sigma})\times\nabla S_{\bar\sigma}$ are in opposite directions, whereas the nonlinear dynamo requires net plasma flow. Using the identities (1.7), on the other hand, we note that particles (σ) and antiparticles ($\bar\sigma$) of the same family ($\sigma,\bar\sigma$) contribute equally to the generation of first-order magnetic fields in a primordial plasma since

$$\sum_{s=\sigma,\bar\sigma} q_s G_{s\nu} = 2\, q_\sigma G_{\sigma\nu},$$
$$\sum_{s=\sigma,\bar\sigma} q_s G_{s\nu}\mathbf{v}_s = q_\sigma G_{\sigma\nu}(\mathbf{v}_\sigma+\mathbf{v}_{\bar\sigma}). \tag{5.8}$$

This remark is especially relevant to the problem of magnetogenesis in the early universe. Conversely, we note from Eq. (5.7) that a time-dependent magnetic field *automatically* generates neutrino-flux vorticity $\nabla\times\boldsymbol{J}_\nu$. Hence, the usual assumption that the neutrino distribution is isotropic [17] appears to be inconsistent with first-order magnetic field generation by first-order collective neutrino-plasma interactions.

B. Magnetic helicity production

Another quantity intimately associated with magnetic field generation is the generation of magnetic helicity

$$H\equiv\int_V \mathbf{A}\cdot\mathbf{B}\, d^3x, \tag{5.9}$$

where V is the three-dimensional volume that encloses the magnetic field lines; to ensure that this definition of magnetic helicity be gauge invariant, we require that $\mathbf{B}\cdot\hat{n}=0$, where $\hat{n}$ is a unit vector normal to the surface ∂V. Magnetic helicity is a measure of knottedness (or flux linkage) in the magnetic field [50]; hence a uniform magnetic field (or more generally a magnetic field that has a global representation in terms of Euler potentials α and β as $\mathbf{B}\equiv\nabla\alpha\times\nabla\beta$) has zero helicity. The production of magnetic helicity is therefore an indication that the spatial structure (and topology) of the magnetic field is becoming more complex. It is expected that this feature in turn plays a fundamental role in the formation of large-scale structure in the universe [2].

The time evolution of the magnetic helicity (5.9) leads to the equation

$$\frac{dH}{dt}=-2c\int_V \mathbf{E}\cdot\mathbf{B}\, d^3x - c\int_{\partial V}(\phi\,\mathbf{B}+\mathbf{E}\times\mathbf{A})\cdot\hat{n}\, d^2x, \tag{5.10}$$

where integration by parts was performed in obtaining the surface term. Taking the integration volume V arbitrarily large (or requiring that $\mathbf{E}$ be parallel to $\hat{n}$ in addition to $\mathbf{B}\cdot\hat{n}=0$), we find that the surface term vanishes and we are left only with the first term in Eq. (5.10). If we now substitute Eq. (5.5) into Eq. (5.10), we obtain

$$\frac{dH}{dt}=-\sum_\sigma\frac{2q_\sigma c}{Q^2}\int_V \mathbf{B}\cdot[\partial_t\mathbf{\Pi}_\sigma-\mathbf{v}_\sigma\times\nabla\times\mathbf{\Pi}_\sigma+\nabla\chi_\sigma + S_\sigma\nabla(\gamma_\sigma^{-1}T_\sigma)]d^3x, \tag{5.11}$$

where $\mathbf{\Pi}_\sigma$ and χ_σ are defined in Eq. (5.6). Since the term $\mathbf{B}\cdot\nabla\chi_\sigma$ can be written as an exact divergence, it does not contribute to the production of magnetic helicity. Furthermore, since temperature gradients along the magnetic field, $\mathbf{B}\cdot\nabla(\gamma_\sigma^{-1}T_\sigma)$, vanish in the absence of dissipative effects the last term in Eq. (5.11) drops out. Hence, magnetic helicity production is governed by the equation

$$\frac{dH}{dt}=-\sum_\sigma\frac{2q_\sigma c}{Q^2}\int_V \mathbf{B}\cdot(\partial_t\mathbf{\Pi}_\sigma-\mathbf{v}_\sigma\times\nabla\times\mathbf{\Pi}_\sigma). \tag{5.12}$$

This equation states that helicity production can occur in the presence of (zeroth-order) nontrivial flows [50] and/or (first-order) nonuniform neutrino flux.

It has been pointed out that magnetic helicity plays an important role in allowing energy to be transferred from small to large scales by a process called inverse cascade. Thus neutrino-flux vorticity leads to the generation of small-scale magnetic fields, first, and then to the production of magnetic helicity. The production of magnetic helicity, on the other hand, converts the small-scale magnetic fields to large-scale magnetic fields, which are expected to play a fundamental role in the problem of structure formation in the early Universe. The magnetic helicity production described by Eq. (5.12) involves a multispecies fluid picture. A more standard description is based on the magnetohydrodynamic (MHD) equations in which plasma flows are averaged over particle species. Future work will proceed by deriving ideal neutrino-MHD equations.

C. Magnetic equilibrium in a magnetized plasma and neutrino fluid

When gravitational effects can be ignored, plasmas can be confined by magnetic fields. Such an equilibrium is established by balancing the (outward) kinetic pressure gradient with the (inward) magnetic pressure gradient. We now investigate how magnetic equilibria are modified by the presence of neutrino fluxes.

The equation for magnetic equilibrium involving magnetic fields associated with neutrino-plasma interactions can be obtained by multiplying Eq. (5.2) with $q_\sigma n_\sigma$ and summing over the charged-particle species only. In a time-independent equilibrium ($\partial/\partial t \equiv 0$) involving a quasineutral plasma (where $\Sigma_\sigma q_\sigma n_\sigma = 0$), a static magnetic field $\mathbf{B}$, and time-independent neutrino fluids, we find the following equilibrium condition:

$$\frac{\mathbf{J}}{c}\times\mathbf{B} = \boldsymbol{\nabla}\cdot\left(\sum_\sigma (n_\sigma \mathbf{v}_\sigma \mathbf{P}_\sigma + \mathbf{I}\, p_\sigma)\right) + \sum_\nu \left[\left(\sum_\sigma n_\sigma G_{\sigma\nu}\right)\nabla n_\nu\right] - \sum_\nu \left[\left(\sum_\sigma G_{\sigma\nu}\frac{n_\sigma \mathbf{v}_\sigma}{c}\right)\times\boldsymbol{\nabla}\times\boldsymbol{J}_\nu\right], \tag{5.13}$$

where $\mathbf{J} \equiv (c/4\pi)\,\boldsymbol{\nabla}\times\mathbf{B} = \Sigma_\sigma q_\sigma \boldsymbol{J}_\sigma$ is the current density flowing in a time-independent magnetized plasma. The first term on the right side of Eq. (5.13) represents the classical term associated with equilibrium in a magnetized plasma. The second and third terms denote first-order neutrino-plasma contributions to magnetic field equilibrium.

Shukla *et al.* [30] derived a similar equilibrium condition with only the electrostaticlike term present on the right side of Eq. (5.13). For a primordial plasma, using Eq. (1.7), we note that the neutrino-induced electrostaticlike term once again vanishes from the magnetic field generation picture. Hence, whereas the second term in Eq. (5.13) vanishes for a primordial plasma, the third term on the right side of Eq. (5.13) does not. Magnetic equilibrium in a primordial neutrino-plasma is thus described by the balance equation

$$\sum_\sigma \frac{\boldsymbol{J}_\sigma}{c}\times\left(q_\sigma \mathbf{B} + \sum_\nu G_{\sigma\nu}\,\boldsymbol{\nabla}\times\boldsymbol{J}_\nu\right) = \boldsymbol{\nabla}\cdot\left(\sum_{s=\sigma,\bar{\sigma}} (n_s \mathbf{v}_s \mathbf{P}_s + \mathbf{I}\, p_s)\right), \tag{5.14}$$

where summation over species on the left side of Eq. (5.14) involves only particle species, while the summation on the right side involves particle and antiparticle species. Once again, neutrino-flux vorticity $\boldsymbol{\nabla}\times\boldsymbol{J}_\nu$ plays a fundamental role in collective neutrino-plasma interactions in the presence of an electromagnetic field.

VI. SUMMARY AND FUTURE WORK

We now summarize our work and discuss future work. The model for collective neutrino-plasma interactions presented in this work is based on the nonlinear dissipationless fluid equations (4.4), (4.9), and (4.18). These equations are derived from a variational principle based on the relativistic covariant Lagrangian density (3.5). An exact energy-momentum conservation law (4.28) is obtained by the Noether method with the energy-momentum tensor for self-consistent collective neutrino-plasma interactions in the presence of an electromagnetic field given by Eq. (4.29). Ponderomotive forces acting on the plasma-neutrino fluids, which are absent from previous works [14,30,31], are given by Eqs. (4.5) and (4.10) [or Eqs. (4.7) and (4.12), respectively]. In Eqs. (5.7) and (5.13), we have demonstrated the crucial role played by neutrino-flux vorticity ($\boldsymbol{\nabla}\times\boldsymbol{J}_\nu$) in the processes of magnetic field generation and magnetic helicity production in neutrino-plasma fluids.

In future work, we plan to further investigate the importance of the neutrino-induced ponderomotive terms associated with neutrino fluxes. For this purpose, it might also be useful to derive ideal neutrino-magnetohydrodynamic equations from Eqs. (4.4), (4.9), and (4.18). Using the mechanisms for magnetic field generation and magnetic helicity production proposed in Eqs. (5.7) and (5.12), respectively, we plan to investigate the problem of magnetogenesis in the early Universe. As another application, we plan to investigate neutrino-plasma three-wave interactions leading to the excitation of various plasma waves in unmagnetized and magnetized plasmas; such transfer processes could be important during supernova explosions.

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APPENDIX: DIFFERENTIAL GEOMETRIC FORMULATION OF CONSTRAINED VARIATIONS

In this Appendix, the geometric interpretation of the constrained variations (3.10), (3.12), (3.14), and (3.16) is given in terms of Lie derivatives along the virtual displacement four-vector $\delta\xi$. Since the variation of a fluid field is only its infinitesimal displacement, all covariant quantities are varied by their Lie derivatives with respect to the virtual displacement four-vector $\delta\xi$. Here, we use the following definition of the Lie derivative on the k-form α along the four-vector $\delta\xi$, denoted $\mathbf{L}_{\delta\xi}\alpha$ [51]:

$$\mathbf{L}_{\delta\xi}\alpha \equiv \mathbf{i}_{\delta\xi}\cdot d\alpha + d(\mathbf{i}_{\delta\xi}\cdot\alpha). \tag{A1}$$

Here, $d\alpha$ is a $(k+1)$-form while $\mathbf{i}_{\delta\xi}\cdot\alpha$ is a $(k-1)$-form representing the contraction of the four-vector $\delta\xi$ with the k-form α. By definition, if $\alpha=\varphi$ is a scalar field (i.e., a zero-form), $\mathbf{i}_{\delta\xi}\cdot\varphi \equiv 0$.

The constrained variation $\delta S = -\delta\xi\cdot\partial S$ for the entropy S [Eq. (3.16)] is consistent with its geometric interpretation as a scalar field:

$$\delta S \equiv -\mathbf{L}_{\delta\xi} S = -\delta\xi\cdot\partial S, \tag{A2}$$

where $\mathbf{i}_{\delta\xi}\cdot S \equiv 0$ and $\mathbf{i}_{\delta\xi}\cdot dS \equiv (\delta\xi\cdot\partial)S$.

The geometric interpretation of the particle flux $J^\alpha \equiv N u^\alpha$ is given as the components of the three-form $J = (1/3!)\epsilon_{\alpha\beta\kappa\lambda} J^\alpha dx^\beta dx^\kappa dx^\lambda$. The constrained variation of the particle-flux four-vector is defined as

$$\delta J \equiv -\mathbf{L}_{\delta\xi} J. \qquad \text{(A3)}$$

Since $dJ \equiv (\partial \cdot J)\,\mathbf{\Omega}$ with the volume four-form $\mathbf{\Omega} \equiv dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, and hence $dJ = 0$ due to the continuity equation, we obtain $\delta J = -d(\mathbf{i}_{\delta\xi} \cdot J)$, or

$$\delta J^\alpha = \partial_\beta (J^\beta \delta\xi^\alpha - J^\alpha \delta\xi^\beta), \qquad \text{(A4)}$$

which is Eq. (3.14) itself. From this variation, one can easily compute the variations of $N = \sqrt{J^\alpha J_\alpha}$ and $u^\alpha = J^\alpha / N$ leading to Eqs. (3.12) and (3.10), respectively.

In Sec. IV D, we consider infinitesimal translations $x^\alpha \to x^\alpha + \delta x^\alpha$ generated by the infinitesimal displacement four-vector δx. Under this transformation, the Lagrangian density $\mathcal{L}$ changes by $\delta\mathcal{L} \equiv -\partial \cdot (\delta x\, \mathcal{L})$. This expression is consistent with the geometric interpretation of $\mathcal{L}$ as a density in four-dimensional space, i.e.,

$$\delta\mathcal{L}\,\mathbf{\Omega} \equiv -\mathbf{L}_{\delta x}(\mathcal{L}\,\mathbf{\Omega}), \qquad \text{(A5)}$$

where $\mathbf{L}_{\delta x}$ is the Lie derivative with respect to δx. Here, using $\mathbf{i}_{\delta x} \cdot d(\mathcal{L}\,\mathbf{\Omega}) = 0$ and

$$d[\mathbf{i}_{\delta x} \cdot (\mathcal{L}\,\mathbf{\Omega})] = d(\mathcal{L}\ \delta x \cdot \omega) \equiv \partial \cdot (\delta x\ \mathcal{L})\ \mathbf{\Omega}, \qquad \text{(A6)}$$

we easily recover Eq. (4.21).

Next, the expression for δA is given in Eq. (4.22). Here, the electromagnetic four-potential A appears as the the components of the one-form $A \cdot dx$. Thus

$$\delta A \cdot dx \equiv -\mathbf{L}_{\delta x}(A \cdot dx). \qquad \text{(A7)}$$

Since $\mathbf{i}_{\delta x} \cdot d(A \cdot dx) = -(\mathsf{F} \cdot \delta x) \cdot dx$ and $d[\mathbf{i}_{\delta x} \cdot (A \cdot dx)] = d(A \cdot dx)$, we easily recover Eq. (4.22) for the four-potential A. We note that the expression $\delta\xi \equiv \mathsf{h} \cdot \delta x$ given in Eq. (4.22) is consistent with the expressions $\delta S = -\mathbf{L}_{\delta\xi} S \equiv -\mathbf{L}_{\delta x} S$ and $\delta J = -\mathbf{L}_{\delta\xi}(J) \equiv -\mathbf{L}_{\delta x}(J)$.

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